Int. J. Heat Mass Transfer. Vol. 36, No. 7, pp. 1799–1806, 1993
Printed in Great Britain

0017-9310/93 $6.00+0.00
Pergamon Press Ltd

Coupling of conduction and forced convection past an impulsively started infinite flat plate

A. POZZI and E. BASSANO
Istituto di Gàsdinamica, Facoltà di Ingegneria, University of Naples, Italy

and

L. DE SOCIO
Dipartimento di Meccanica ed Aeronautica, University of Rome, Italy

(*Received* 16 *January* 1992 *and in final form* 23 *July* 1992)

Abstract—In this paper the unsteady coupling of conduction and convection for a thin body in a high speed stream is considered. The body is modeled as a strip on one side of which the fluid is impulsively started in motion whereas on the other side two different thermal boundary conditions are given: constant temperature or vanishing heat flux. In the first part of the analysis an approximate solution of the energy equation in the solid enables us to obtain relations between the temperature and the heat flux at solid–fluid interface which are more accurate than those currently used in the literature. In the second part the exact solution of the two problems that arise from the coupling of the thermofluid-dynamic equations and the relations between the temperature and its derivatives at the interface are presented.

1. INTRODUCTION

PROBLEMS which arise from the coupling of conduction in a solid body and of conduction and convection in a unsteady stream flowing around it present noticeable interest in applications. This is the case, for example, in aerospace situations where a fast flow interacts with the external surface of a usually thin wall, the other surface of which is kept adiabatic or in isothermal conditions.

An adiabatic condition is also realized, of course, in cases when a plane of symmetry is present. A model for these situations corresponds to the assumption that the solid body behaves as a flat plate from the fluid-dynamic point of view.

The thermofluid-dynamic field along an impulsively started flat plate with vanishing thickness, studied at the beginning of this century [1], is described by very simple functions.

When one considers the motion of a flat plate with a non-zero thickness, the coupled thermal field in the solid and thermofluid-dynamic field in the fluid must be determined. This problem can be simplified when it is possible to adopt an approximate solution of the energy equation in the solid and thereafter to solve the thermofluid-dynamic equations in the fluid with a suitable boundary condition for the energy equation at the solid–fluid interface. This condition was obtained (while studying the thermal entrance region of a duct) by Sparrow and De Farias [2] and by Sucec [3], by means of an energy balance on an element of axial extension dx of the solid body by assuming a mean value of the temperature throughout its thickness.

In this paper we present a different and more accurate thermal condition at the solid–fluid interface and we solve the problem for the suddenly started flat plate, also taking into account the heat produced by friction.

2. TEMPERATURE IN THE SOLID

The temperature field in a two-dimensional homogeneous solid is governed by the heat-conduction equation

$$T_{t'} = d\Delta_2 T \tag{1}$$

where t' is the dimensional time and d is the thermal diffusivity $d = \lambda/\rho c$, and λ, ρ and c are the thermal conduction coefficient, the density and the specific heat.

Let us consider a strip of thickness b along a y' axis and characteristic length L along an x' axis. Then, in dimensionless form, equation (1) can be written as

$$T_\tau = T_{YY} + (b^2/L^2)T_{XX} \tag{2}$$

where $X = x'/L$, $Y = y'/b$, and the non-dimensional time is $\tau = t'd/b^2$.

In particular, if temperature is supposed not to depend on the X variable equation (2) reduces to

$$T_\tau = T_{YY}. \tag{3}$$

As a first step we consider the temperature distribution in an infinite strip, when governed by equation (3) for two types of boundary conditions. Case (a) corresponds to $T = T_w(\tau)$ on the upper side and $T = T_c = \text{const.}$ on the lower side. In case (b)

NOMENCLATURE

b	plate thickness
c	specific heat of the solid
d	thermal diffusivity
L	reference length in x' direction
M	Mach number
p	first coupling parameter
Pr	Prandtl number
Re	Reynolds number
t	dimensionless time for the fluid
t'	dimensional time
T	dimensionless temperature referred to the asymptotic one
T_p	particular solution of the unhomogeneous problem
T^+	solution of the homogeneous problem
t_{fs}	second coupling parameter
u	dimensionless velocity component in x' direction
U	Stewartson–Dorodnitzin velocity component in x' direction
v	dimensionless velocity component in y' direction
V	Stewartson–Dorodnitzin velocity component in y' direction
x	fluid dimensionless abscissa
X	solid dimensionless abscissa
x'	dimensional abscissa
y	fluid dimensionless ordinate
Y	solid dimensionless ordinate
y'	dimensional ordinate.

Greek symbols

γ	specific heat coefficients ratio
η	Stewartson–Dorodnitzin ordinate
θ	Laplace transform of T^+
λ	thermal conductivity
μ	absolute viscosity
ν	kinematic viscosity
ξ	Stewartson–Dorodnitzin abscissa
ρ	density
τ	dimensionless time for the solid.

Subscripts

e	quantities evaluated at the lower wall
f	quantities evaluated in the fluid
0	quantities evaluated at $t = 0$
s	quantities evaluated in the solid
w	quantities evaluated at the upper wall
∞	quantities evaluated at infinity.
case a	isothermal case
case b	adiabatic case.

$T = T_w(\tau)$ on the upper side and $T_Y = 0$ on the lower side (Fig. 1).

If equation (3) is written for the upper side of the strip

$$T_{w\tau} = T_{YY,w} \quad (4)$$

its right-hand side can be approximated by

$$T_{YY,w} = T_{Y,w} - T_{Y,e} \quad (5)$$

where 'e' stands for the lower external side of the strip. Furthermore if one assumes

$$T_{Y,e} = T_w - T_e \quad (6)$$

then one obtains the following approximated relations

$$\text{(a) } T_{w\tau} = T_{Y,w} - T_w + T_e\,; \quad \text{(b) } T_{w\tau} = T_{Y,w} \quad (7)$$

for cases (a) and (b), respectively. Equations (7) and the relations which express the continuity of the temperature and of the heat flux at the fluid–solid interface were usually adopted in the literature (see, e.g. Sparrow and De Farias [2], Sucec [3], Kim and Ozisik [4]) as boundary conditions for the energy equation in conjugated heat transfer problems. However, simple conditions more accurate than equations (7) can be obtained by integration of equation (3) with respect to Y between 0 and 1. In particular, since

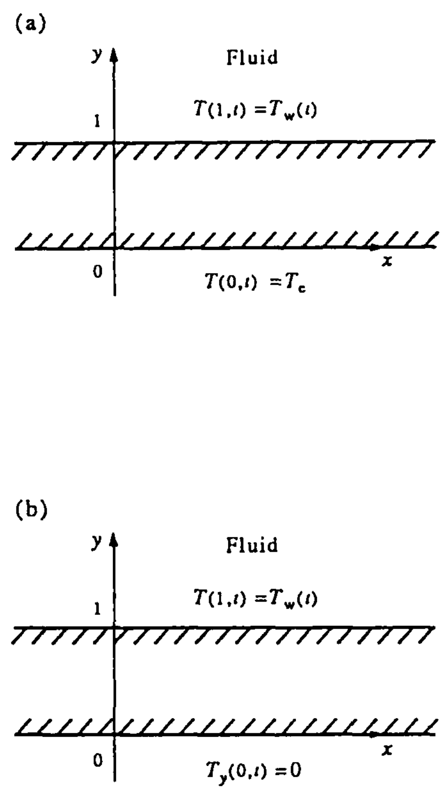

FIG. 1. Boundary conditions in the solid.

$$\frac{\partial}{\partial \tau}\int_0^1 T\,\mathrm{d}Y = T_{Y,\mathrm{w}} - T_{Y,\mathrm{e}} \tag{8}$$

after multiplying equation (3) by Y and integrating one has

$$\frac{\partial}{\partial \tau}\int_0^1 YT\,\mathrm{d}Y = T_{Y,\mathrm{w}} - T_{\mathrm{w}} + T_{\mathrm{e}}. \tag{9}$$

For $T_{Y,\mathrm{e}} = 0$ equation (8) immediately gives

$$\frac{\partial}{\partial \tau}\int_0^1 T\,\mathrm{d}Y = T_{Y,\mathrm{w}}. \tag{10}$$

Now we approximate the integral relations (9) and (10) (rather than differential relations) by assuming a suitable linear expression either for T or for T_Y, respectively.

Case (a)

In order to calculate the integral at the left-hand side of equation (9) we assume $T = T_{\mathrm{e}} + (T_{\mathrm{w}} - T_{\mathrm{e}})Y$. Then, equation (9) leads to

$$T_{\mathrm{w}\tau} = 3(T_{Y\mathrm{w}} - T_{\mathrm{w}} + T_{\mathrm{e}}). \tag{11}$$

Case (b)

In this case the integral at the left-hand side of equation (10) is evaluated letting $T_Y = T_{Y\mathrm{w}}Y$ and one has

$$3T_{\mathrm{w}\tau} - T_{Y\mathrm{w}\tau} = 3T_{Y\mathrm{w}}. \tag{12}$$

Equations (11) and (12) are more accurate than equation (7) since no approximations were introduced in the evaluation of differential expressions but only at an integral level. The accuracy of these results will be checked by comparison with some exact solutions. Simple exact solutions of equation (3) with the boundary conditions corresponding to cases (a) and (b) can be obtained in separable form. In the first situation one has $T - T_{\mathrm{e}} = \exp(-h^2\tau)\sin Yh$, where h is an arbitrary constant.

By assuming $h = \pi/2$ our condition (11) leads to an approximate value for $\pi^2/4$ equal to 3, whereas equation (7a) provides a result equal to 1. By assuming $h = \pi/4$ equation (11) leads to $0.645 = 0.616$, whereas equation (7a) gives $0.215 = 0.616$. A second solution is $T - T_{\mathrm{e}} = \mathrm{e}^{H^2 t}\,\mathrm{Sin}\,hY$, where Sin stands for the hyperbolic sine, and h is an arbitrary constant. For $h = 1$ equation (11) gives an approximate value for 1.175 equal to 1.104, whereas equation (7a) yields 0.378.

Similar comparisons can be performed in case (b). Exact solutions can be obtained from the previous ones by replacing sin (or Sin) by cos (or Cos). For $T = \exp(-\tau\pi^2/16)\cos(Y\pi/4)$, equation (12) approximates 2.97 by 3 whereas equation (7b) approximates 0.785 by 1.

3. EQUATIONS AND BOUNDARY CONDITIONS FOR THE THERMOFLUID-DYNAMIC FIELD

The thermofluid-dynamic field in the flow close to the wall is assumed to be governed by the laminar boundary layer equations, which in non-dimensional form when the outer velocity is constant may be written as

$$\rho_t + (\rho u)_x + (\rho v)_y = 0 \tag{13}$$

$$[\rho(u_t + uu_x + vu_y) = (\mu u_y)_y]$$

$$[\rho(T_t + uT_x + vT_y) = \frac{1}{Pr}(\lambda T_y)_y + (\gamma - 1)M^2\mu u_y^2]. \tag{14}$$

We take now as reference lengths L and $L/Re^{1/2}$ along the x and y directions, respectively, where $Re = u_\infty L/\nu_\infty$ and ν_∞ is the asymptotic kinematic viscosity of fluid; the reference time is L/u_∞ and temperature, density, thermal conductivity coefficient and the reference velocity are the asymptotic ones. M and Pr are the Mach and Prandtl numbers.

By assuming that μ and λ vary linearly with temperature the boundary layer equations reduce to the incompressible form by the Stewartson–Dorodnitzin transformation and equation (13) may be solved independently of equation (14). Let

$$\xi = x;\quad \eta = \int_0^y \rho\,\mathrm{d}y;\quad U = u;\quad V = \rho v + u\eta_x + \eta_t. \tag{15}$$

Equations (13)–(15) become

$$U_\xi + V_\eta = 0 \tag{16}$$

$$U_t + UU_\xi + VU_\eta = U_{\eta\eta} \tag{17}$$

$$T_t + UT_\xi + VT_\eta = T_{\eta\eta}/Pr + (\gamma - 1)M^2U_\eta^2 \tag{18}$$

associated with the boundary conditions

$$U(\xi, 0, t) = V(\xi, 0, t) = 0;\qquad T(\xi, \eta, t^-) = 1 \tag{19}$$

$$U(\xi, 0, t^+) = T(\xi, \infty, t^+) = 1;\quad U(\xi, \eta, t^-) = 0 \tag{20}$$

where t^+ and t^- stay for $t > 0$ and $t < 0$, respectively.

At the interface ($\eta = 0$) one has that the temperature of the fluid equals the temperature of the solid. Moreover the heat flux is continuous, i.e.

$$\lambda_s^{(\mathrm{d})}T_{y',s}^{(\mathrm{d})} = \lambda^{(\mathrm{d})}T_{y'}^{(\mathrm{d})} \tag{21}$$

where (d) denotes dimensional quantities. The right-hand side of this equation may be written in terms of non-dimensional quantities as $\lambda_\infty T_\infty T_\eta Re^{1/2}L$ and hence equation (21) gives the last condition for the energy equation

$$T_{Y,\mathrm{w}} = (b\,Re^{1/2}\,\lambda_\infty/L\lambda_s)T_{\eta,\mathrm{w}} \tag{22}$$

which, after equations (11) and (12), can be written as

Case (a)

$$T_{wt} = 3t_{fs}(pT_{\eta,w} - T_w + T_e) \quad (23)$$

Case (b)

$$(3/p)T_{wt} - T_{\eta,wt} = 3t_{fs}T_{\eta,w} \quad (24)$$

where $t_{fs} = L\,d/u_\infty b^2$ and $p = b\,Re^{1/2}\,\lambda_\infty/L\lambda_s$.

4. THERMOFLUID-DYNAMIC FIELD PAST AN INFINITE STRIP

4.1. *Velocity profile*

Since the thermofluid-dynamic field does not depend on x, for an infinite strip equations (16) and (17) lead to the well known solution

$$U = \text{erf}\,(\eta/2t^{1/2}) \quad (25)$$

where erf denotes the error function (see Abramovitz and Stegun [5]).

The velocity profiles in the physical plane will then be obtained by solving the energy equation (18) because the density distribution must be evaluated in order to recover the physical coordinate y from η through equation (15b).

4.2. *Solution of energy equation*

For an infinite strip equation (18) reduces to

$$T_t - T_{\eta\eta}/Pr = (\gamma-1)M^2U_\eta^2 \quad (26)$$

and the right-hand side is equal to $(\gamma-1)M^2\exp(-\eta^2/2t)/\pi t$.

To solve equation (26) in cases (a) and (b) we put $T = 1 + T^+ + T_p$, where T_p is a particular solution of equation (26), for $T(\infty, t) = T(\eta, 0)$. A similar solution for $\xi = \eta/2t^{1/2}$, can be obtained in the form

$$T_p = M^2\left\{C - 2[(\gamma-1)(Pr/\pi)^{1/2}]\int_0^\xi f\,d\xi + [(\gamma-1)/K]\,\text{erfc}\,(Pr^{1/2}\xi)\,\text{erfc}\,(K\xi\,Pr^{1/2})\right\} \quad (27a)$$

where

$$f = \exp\left[-(K\xi\,Pr^{1/2})^2\right]\text{erfc}\,(\xi\,Pr^{1/2}),$$

$$K^2 = 2/Pr - 1$$

and

$$C = 2(\gamma-1)(Pr/\pi)^{1/2}\int_0^\infty f\,d\xi = [2(\gamma-1)/\pi]\arctan K/K \quad (27b)$$

For $Pr > 2$ K is imaginary but equations (27a) and (27b) are still valid and can be written as

$$T_p = M^2\left\{C - 4\left[(\gamma-1)\delta^2\int_0^{\xi Pr^{1/2}/\delta} f_2(s)\,ds\right]\Big/\pi\right\}$$

$$C = \left[4(\gamma-1)\delta^2\int_0^\infty f_2(s)\,ds\right]\Big/\pi = 2(\gamma-1)\arctan h\,\delta/\delta\pi$$

where

$$\delta^2 = Pr/(Pr-2) = -1/K^2$$

and

$$f_2(s) = e^{-(\delta s)^2}\int_0^s e^{t^2}\,dt.$$

It results $T_{pw} = T_p(0,t) = CM^2$, $T_p(\eta, 0) = 0$, $T_{p\eta,w}(0,t) = 0$, $T_{pt,w}(0,t) = 0$. For $Pr = 1$ one has

$$T_p = (\gamma-1)M^2(1 - \text{erf}^2\,\xi)/2 \quad (27c)$$

and for $Pr = 2$

$$C = 2(\gamma-1)/\pi,$$

$$T_p = CM^2 + 2(\gamma-1)M^2(e^{-2\xi^2} - 1)/\pi = 2(\gamma-1)M^2\,e^{-2\xi^2}/\pi.$$

T^+ is obtained from $T_t^+ - T_{\eta\eta}^+/Pr = 0$ with $T^+(\infty, t) = T^+(\eta, 0) = 0$ and with the further conditions (23) or (24), which in terms of T^+ give

Case (a)

$$T_{wt}^+ = 3t_{fs}(pT_{\eta w}^+ - T_w^+ + T_e - 1 - CM^2) \quad (28a)$$

Case (b)

$$(3/p)T_{wt}^+ - T_{\eta wt}^+ - 3t_{fs}T_{\eta w}^+ = 0. \quad (28b)$$

Let $\theta(s, \eta)$ be the Laplace transform, with respect to t, of T^+, which is governed by $s\theta - \theta_{\eta\eta}/Pr = 0$. The solution convergent at infinity is $\theta = A\exp(-Pr^{1/2}s^{1/2}\eta)$, where A is determined by the transform of equation (28a) or (28b).

Case (a)

$$s\theta(s,0) + CM^2 = 3t_{fs}[p\theta_\eta(s,0) - \theta(s,0) + (T_e - 1 - CM^2)/s] \quad (29a)$$

Case (b)

$$(3s/p)\theta(s,0) - s\theta_\eta(s,0) + T_{\eta_{w,0}} - 3t_{fs}\theta_\eta(s,0) = 0. \quad (29b)$$

As $\theta(s,0) = A$ and $\theta_\eta(s, 0) = -A\,Pr^{1/2}\,s^{1/2}$ we find the following values for A:

Case (a)

$$A = [3t_{fs}(T_e - 1 - M^2C)/s - M^2C][1/(s^{1/2} - s_1) - 1/(s^{1/2} - s_2)]/(s_1 - s_2)$$

where

$$s_{1,2} = -3t_{fs}p\,Pr^{1/2}/2 \pm (9t_{fs}^2p^2\,Pr/4 - 3t_{fs})^{1/2}.$$

Case (b)

$$A = -T_{\eta_{w,0}}[1/(s^{1/2}-s_1)-1/(s^{1/2}-s_2)]/[(s_1-s_2)(Pr\,s)^{1/2}]$$

where

$$s_{1,2} = -3/(2p\,Pr^{1/2}) \pm (9/(4p^2\,Pr)-3t_{fs})^{1/2}.$$

In both cases it is $Re(s_{1,2}) < 0$. Since equation (28a) is not compatible with $T^+_{wo} \neq 0$ the wall temperature in case (b) is discontinuous.

4.3. *Temperature and heat transfer at the interface*

The knowledge of the expressions of C and A enables one to calculate the temperature and the heat transfer at the interface. To better show the influence on T_w and $T_{\eta,w}$ of the thickness b, we write the expressions of these functions together with those for the flat plate of zero thickness. Since $T_w = 1+T_{pw}+T^+_w$ and $T_{\eta,w} = T^+_{\eta,w}$, we have (see Erdelyi *et al.* [6]) in case (a)

$$T_w = T_e + [(T_e-1-M^2C)(s_2f_1-s_1f_2) - M^2C(s_1f_1-s_2f_2)]/(s_1-s_2)$$

$$T_{\eta w} = Pr^{1/2}\{M^2C/(\pi t)^{1/2}+[3t_{fs}(T_e-1-M^2C)(f_2-f_1) + M^2C(s_1^2f_1-s_2^2f_2)]/(s_1-s_2)\} \quad (30)$$

where $f_i = \exp(s_i^2t)\,\mathrm{erfc}\,(-s_it^{1/2})$. Whereas for $b = 0$

$$T_w = T_e; \quad T_{\eta_w} = (M^2C+1-T_e)(Pr/\pi t)^{1/2}.$$

Analogously, in case (b)

$$T_w = 1+CM^2-T^+_{\eta_{w0}}(f_1-f_2)/(s_1-s_2)\,Pr^{1/2}$$
$$T_{\eta_w} = T^+_{\eta_{w,0}}(s_1f_1-s_2f_2)/(s_1-s_2)$$

where $I_{\eta w,0} = T^+_{\eta w}(0)$ is given by the energy equation in the integrated form (10). This equation shows that

$$\int_0^1 T\,\mathrm{d}Y \simeq (T_w - T_{yw}/3)$$

must be continuous. Therefore $T^+_{\eta_{w,0}} = 3M^2C/p$. For comparison, when $b = 0$; $T_w = 1+\mathrm{C}M^2$, $T_{\eta_w} = 0$.

4.4. *Temperature profiles*

The temperature profiles are given by

$$T(\eta, t) = 1+T_p+T^+ \quad (31)$$

where T_p is given by equation (27a). In the various cases the expressions for T^+ follow.

Case (a)

$$T^+(\eta, t) = (T_e-1-M^2C)[\mathrm{erfc}\,(\xi\,Pr^{1/2}) + (s_2g_1-s_1g_2)/(s_1-s_2)] \quad (32)$$
$$-CM^2(s_1g_1-s_2g_2)/(s_1-s_2) \quad (32)$$

where

$$g_i = \mathrm{e}^{-\xi^2 Pr+(\xi Pr^{1/2}-s_it^{1/2})^2}\,\mathrm{erfc}\,(\xi\,Pr^{1/2}-s_it^{1/2})$$

and for $b = 0$, $T^+(\eta, t) = (T_e-1-CM^2)\,\mathrm{erfc}\,(\xi\,Pr^{1/2})$.

Case (b)

$$T^+(\eta, t) = 3CM^2(g_2-g_1)/p\,Pr^{1/2}(s_1-s_2) \quad (33)$$

and for $b = 0$, $T^+ = 0$.

4.5. *Analysis of the results*

The coupling of conduction in the solid and conduction and convection in the fluid in the investigated problem introduces two new non-dimensional parameters $t_{fs} = Ld/u_\infty b^2$ and $p = Re^{1/2}b\lambda_\infty/\lambda_sL$ (in addition to the Prandtl, Reynolds and Mach numbers). In order to analyze the role of these new parameters we list in Table 1 the expressions of the temperature and heat transfer at the wall for $t \to 0$ and $t \to \infty$ together with the corresponding ones for $b = 0$. In table 1 $\alpha = (Pr/\pi)^{1/2}$, $\beta = 1+CM^2-T_e$, $\delta = CM^2/3t_{fs}p(Pr\,\pi)^{1/2}$.

In case (a) the wall temperature T_w is continuous at $t = 0$ because the energy equation in the solid, in the form (9), requires the continuity of $\int_0^1 YT\,\mathrm{d}Y \simeq T_w/3$. Therefore the leading term for T_w is the unperturbed one, in contrast with the case $b = 0$ where the discontinuity of the temperature is imposed by the boundary conditions. The second term shows that T_w

Table 1. Wall temperature and heat flux behaviour for $t \to 0$ and $t \to \infty$

Case (a)

	$t \to 0$		$t \to \infty$	
	$b \neq 0$	$b = 0$	$b \neq 0$	$b = 0$
T_w	$1+6pt_{fs}M^2C\alpha t^{1/2}$	T_e	$T_e+\beta p\alpha/t^{1/2}$	T_e
$T_{\eta w}$	$M^2C(\alpha/t^{1/2}-3t_{fs}p\,Pr)$	$\beta\alpha/t^{1/2}$	$\beta\alpha[1-(p^2\,Pr-(1+M^2C/\beta)/3t_{fs})/2t]/t^{1/2}$	$\beta\alpha/t^{1/2}$

Case (b)

	$t \to 0$		$t \to \infty$	
	$b \neq 0$	$b = 0$	$b \neq 0$	$b \equiv 0$
T_w	$1+M^2C-6M^2Ct^{1/2}/p(Pr\,\pi)^{1/2}$	$1+M^2C$	$1+M^2C-3\delta/t^{1/2}$	$1+M^2C$
$T_{\eta w}$	$3M^2C(1-6t^{1/2}/p(Pr\,\pi)^{1/2})/p$	0	$3\delta/2pt_{fs}t^{3/2}$	0

Table 2. Physical ordinate for $Pr = 1$

	$t \to 0$		$t \to \infty$	
	$b \neq 0$	$b = 0$	$b \neq 0$	$b = 0$
y (case (a))	$\eta + 2CM^2(2^{1/2}-1)(t/\pi)^{1/2}$	$\eta + 2CM^2(2t/\pi)^{1/2}$	$\eta[T_e + \beta(\eta/2 + 3t_{fs}p)/(\pi t)^{1/2}]$	$\eta[T_e + \beta\eta/2(\pi t)^{1/2}]$
y (case (b))	$\eta + 2CM^2(2t/\pi)^{1/2}$	$\eta + 2CM^2(2t/\pi)^{1/2}$	$\eta\{1 + CM^2[1 - 1/t_{fs}p(\pi t)^{1/2}]\}$	$\eta(1 + CM^2)$

increases as $M^2 t_{fs} p\, Pr^{1/2} C(Pr) t^{1/2}$, the heat transfer T_{η_w} is infinite at $t = 0$, its leading term depending on M^2 but not on the imposed temperature, as for $b = 0$.

The asymptotic behaviors of T_w and T_{η_w} are the same as for $b = 0$, the second order term of T_w which depends on p only whereas the one in T_{η_w} depends also on t_{fs}.

In case (b), for $t \to 0$ the temperature T_w is discontinuous at 0^+ as in the case $b = 0$, whereas the heat transfer T_{η_w} does not vanish. Both T_w and T_{η_w} vary linearly at the second order with the ratio M^2C/p.

Once again the asymptotic behaviors for $t \to \infty$ are the same as for the case $b = 0$.

When $Pr = 1$ the temperature can be expressed in closed form (see equations 27(c), 31–33) as also the physical coordinate y

$$y = \int_0^\eta T(\eta, t)\, d\eta = \eta(1 + CM^2)$$

$$- CM^2 2t^{1/2}[\xi \operatorname{erf}^2 \xi + 2\, e^{-\xi^2} \operatorname{erf} \xi/\pi^{1/2}$$

$$- (2/\pi)^{1/2} \operatorname{erf}(\xi 2^{1/2})] + \int_0^\eta T^+\, d\eta$$

with

$$\int_0^\eta T^+\, d\eta$$

given by

Case (a)

$$\int_0^\eta T^+\, d\eta = 2t^{1/2}(T_e - 1 - CM^2)[\xi \operatorname{erfc} \xi$$

$$+ (1 - e^{-\xi^2})/\pi^{1/2}] + [(T_e - 1 - CM^2)(s_2 h_1/s_1 - s_1 h_2/s_2)$$

$$- CM^2(h_1 - h_2)]/(s_1 - s_2)$$

Case (b)

$$\int_0^\eta T^+\, d\eta = -M^2C(s_2 h_1 - s_1 h_2)/t_{fs}p(s_1 - s_2)$$

where

$$h_i = \{[-e^{-\xi^2 + (\xi - s_i t^{1/2})^2} \operatorname{erfc}(\xi - s_i t^{1/2})]_0^\xi - \operatorname{erf} \xi\}.$$

The expression of y for $t \to 0$ and $t \to \infty$, in case (a) and (b) for $Pr = 1$ is reported in Table 2 (up to power $t^{1/2}$ and $t^{-1/2}$, respectively).

Some results, related to $M = 3$, $Pr = 1$ and $\gamma = 1.4$, are displayed in Figs. 2–6; p and t_{fs} vary in the range (0.01, 1). In case (a) $T_e = 1.3$. In Figs. 2 and 3 the curves of T_w for cases (a) and (b) are drawn. In Figs. 4 and 5 the temperature profiles are drawn in the physical plane for cases (a) and (b). In Fig. 6 the velocity profiles for case (b) are drawn always in the

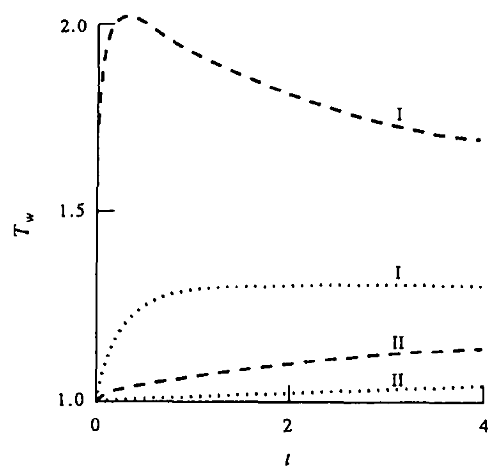

Fig. 2. Temperature at the wall, case (a) (- - - $p = 1$, ··· $p = 0.01$, I $t_{fs} = 1$, II $t_{fs} = 0.01$).

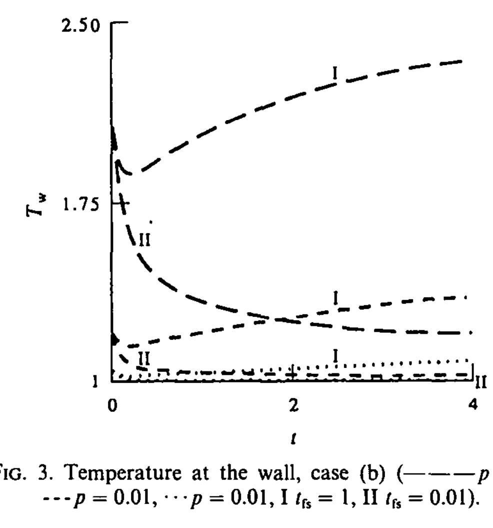

Fig. 3. Temperature at the wall, case (b) (——— $p = 1$, - - - $p = 0.01$, ··· $p = 0.01$, I $t_{fs} = 1$, II $t_{fs} = 0.01$).

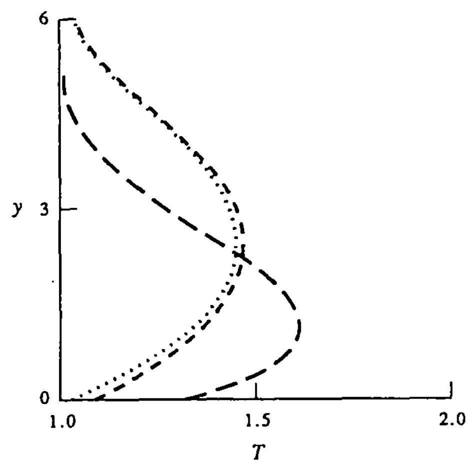

FIG. 4. Temperature profiles, $t_{fs} = 0.01$, case (a) ($\cdots p = 0.01$, $--- p = 1$, $——— b = 0$).

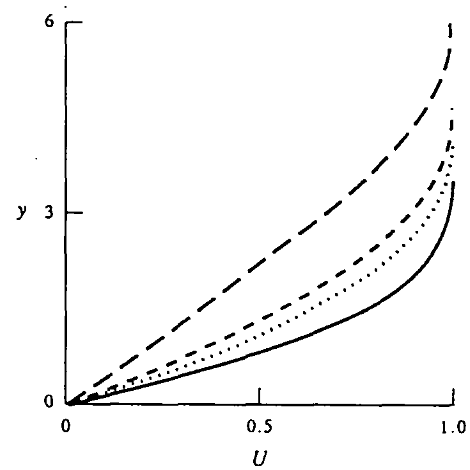

FIG. 6. Velocity profiles, $t_{fs} = 0.01$, case (b) ($\cdots p = 0.01$, $--- p = 1$, $——— b = 0$, $—— M = 0$).

physical plane. In case (a) the velocity profiles corresponding to the same values of p and t_{fs} of case (b) fall on the curve $p = 1$ of Fig. 6. In Figs. 4–6 the profile corresponding to $b = 0$ is also reported.

5. CONCLUDING REMARKS

We have studied the thermofluid-dynamic field resulting from the coupling of laminar forced convection along and conduction inside a suddenly started flat plate, under two different thermal boundary conditions (cases (a) and (b) of Fig. 1).

5.1. *Wall temperature distribution*

This analysis, performed by finding accurate approximate solutions for the energy equations in the fluid, enabled us to evaluated the influence of the coupling parameters, P and t_{fs}, on the field. Some results are reported in tables and diagrams for $M = 3$ and $Pr = 1$, in case (a) $Te = 1.3$. We note that in case (a) the wall temperature T_w grows either with increasing p at t_{fs} fixed or with increasing t_{fs} for fixed p. For t vanishing T_w increases the more rapidly, the larger are the two parameters, whereas for t not too small T_w can show a maximum, as shown in Fig. 2. In case (b), for p and t_{fs} fixed, T_w has an initial steep decrease, more pronounced when p is small, and shows a minimum.

At t_{fs} fixed T_w increases as p increases; at p fixed, with decreasing t_{fs}, the minimum decreases and moves towards larger values of t (Fig. 3).

5.2. *Temperature profiles*

Let us consider now the influence of p and t_{fs} on the temperature profiles; we assume $t = 1$. In case (a) (Fig. 4) the temperature profiles for large values of t_{fs} ($> 10^2$) and small values of p ($< 10^{-2}$) coincide with the one corresponding to $b = 0$ (for $b \to 0, p \to 0, t_{fs} \to \infty$) which has a maximum at y equal to approximately 1.

With increasing p the temperature grows markedly, the point of maximum moves towards the wall and the asymptotic temperature is 'reached' at ever increasing values of y.

For values of t_{fs} decreasing below order one the temperature decreases and the profiles corresponding to different p tend towards a limit profile having a maximum greater than that of $b = 0$ (Fig. 4).

In case (b), for $t_{fs} \ll 1$, the temperature increases strongly with p and the point of maximum moves towards the wall (Fig. 5). For t_{fs} of order one the temperature profiles do not change appreciably with respect to those for $t_{fs} \ll 1$, the temperature increases slightly only for p of order one.

When $t_{fs} \gg 1$, the temperature increases with

FIG. 5. Temperature profiles, $t_{fs} = 0.01$, case (b) ($\cdots p = 0.01$, $--- p = 1$, $——— b = 0$).

increasing p and the temperature profiles tend to the limit one corresponding to $p \to \infty$.

5.3. *Velocity profiles*

Let us now consider the influence of p and t_{fs} on the velocity profiles, again for $t = 1$. We remember that in the transformed plane the velocity profile does not depend on the temperature and U is always given by equation (25). In the physical plane the velocity profile is obtained by expressing η in terms of y.

In case (a), $\eta < y$, y/η increases with increasing p and the velocity profile tends to an upper limit profile for $p \to \infty$ (for any t_{fs}). For $t_{fs} \geqslant 1$ the velocity profiles lie near to that corresponding to $b = 0$ for small values of p and lie near the limit one for high values of p, while for $t_{fs} \ll 1$ the velocity profiles are all near the limit one.

In case (b), $\eta > y$. For $t_{fs} \leqslant 1$ the velocity profiles for $p \gg 1$ lie near to that for $b = 0$, while the velocity profiles for $p \ll 1$ lie near to a lower limit profile (for $p \to 0$). For $t_{fs} \gg 1$ the velocity profiles tend to that corresponding to $b = 0$; those corresponding to smaller values of p become the nearest to it.

5.4. *Influence of finite thickness*

Finally, we note that the presence of a finite thickness can influence the discontinuity of temperature and heat flux at the wall at $t = 0$.

In case (a) the finiteness of the plate thickness leads to a continuous wall temperature at $t = 0$ since the temperature T_e is no more imposed as $\eta = 0$ as for $b = 0$. The heat flux goes to infinity for $t \to 0$ as $t^{-1/2}$ but its leading term depends on M and Pr and not on T_e as for $b = 0$.

In case (b), in the presence of a finite thickness, the wall temperature is discontinuous as $t = 0$ as when $b = 0$, the heat flux is finite for $t \to 0$, while for $b = 0$ it vanishes. It is interesting to note that in case (b) the solution that satisfies all the boundary conditions (19)–(20) is defined by the arbitrary constant $T_{\eta w}(0)$.

In order to find the value of $T_{\eta w}(0)$ we have imposed the energy conservation inside the solid, expressed by the continuity, with respect to time at $t = 0$, of the integral in equation (10). By employing this procedure we have obtained $T_{\eta w}(0) = 3CM^2/p$.

Acknowledgement—This work was supported by MURST (Italian Ministry of University and Research).

REFERENCES

1. Rayleigh, On the motion of solid bodies through viscous liquids, *Phil. Mag.* **21,** 697–711 (1911).
2. E. M. Sparrow and F. N. De Farias, Unsteady heat transfer in duct with time-varying inlet temperature and participating walls, *Int. J. Heat Mass Transfer* **11,** 837–853 (1968).
3. J. Sucec, Exact solution for unsteady conjuated heat transfer in the thermal entrance region of a duct, *J. Heat Transfer* **109,** 295–299 (1987).
4. W. S. Kim and M. N. Ozisik, Conjugated laminar forced convection in ducts with periodic variation of inlet temperature, *Int. J. Heat Fluid Flow* **11,** 311–320 (1990).
5. M. Abramovitz and I. Stegun, *Handbook of Mathematical Functions.* Dover, New York (1968).
6. A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Bateman Manuscript Project.* McGraw-Hill, New York (1954).